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Linear Algebra and its Applications 295 (1999) 117–132

LINEAR ALGEBRA AND ITS APPLICATIONS

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Nearly defective complex eigenvalues in Bessel matrices

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Received 26 March 1998; accepted 7 April 1999

Submitted by F. Uhlig

Abstract

It is known that an $n \times n$ Bessel matrix T_n exhibits simple well-separated complex eigenvalues which are getting more and more ill-conditioned for increasing n. The aim of this paper is to study the behavior of such eigenvalues, which are simple in theory, but in practice become defective using finite precision. To this end we provide a sharp estimate of the distance of T_n to the nearest defective matrix, when n increases. © 1999 Elsevier Science Inc. All rights reserved.

Keywords: Defective eigenvalues; Defective matrices; Bessel matrices

1. Introduction

We consider the case of ill-conditioned eigenvalues of the $n \times n$ real matrix defined as follows:

$$T_n = \begin{bmatrix} -1 & \sigma_1 & & & \\ -\sigma_1 & 0 & \sigma_2 & & \\ & -\sigma_2 & \cdot & \cdot & \\ & & \cdot & \cdot & \sigma_{n-1} \\ & & & -\sigma_{n-1} & 0 \end{bmatrix}, \tag{1}$$

where $\sigma_h = -1/(4h^2 - 1)^{1/2}$, $h = 1, 2, \ldots, n-1$, $n \in \mathbb{N}^+$.

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0024-3795/99/$ – see front matter
PII: S0024-3795(99)00089-0

T_n has a set of simple complex eigenvalues.

From an analytical point of view, it is known that the eigenvalues are zeros of Bessel polynomials [20] and lie along a curve in $\mathbb{C}$ in a symmetric way with respect to the real axis, in the left-half of the Gaussian plane [1]. The curves vary with n and quickly tend to the origin. However if we "normalize" them so that $\{\pm i\}$ become the endpoints, then the curves asymptotically tend to a limiting curve (for details see [1, p. 477]). When n is odd T_n has one single real eigenvalue and pairs of complex conjugate eigenvalues, whereas when n is even T_n has only pairs of complex conjugate eigenvalues.

From a numerical point of view, it is known [14,15,17] that for increasing n the individual condition numbers of the eigenvalues of T_n increase very quickly. In particular this is the case of the eigenvalues closest to the real axis (further details will be given in Section 2). Actually when $n \geqslant 25$ (in double precision arithmetic) the eigenvalues become so ill-conditioned that any graphical representation shows a peculiar bifurcation of such computed eigenvalues. They will simply not lie along a curve any longer. In Fig. 1 such behavior for T_n, $n = 30$, is reported (the eigenvalues are computed by means of MATLAB **eig**).

The matrix T_n is a challenging test matrix, which allows us to study the problem of nearly defective complex eigenvalues, in a special case when eigenvalues lie along a single curve.

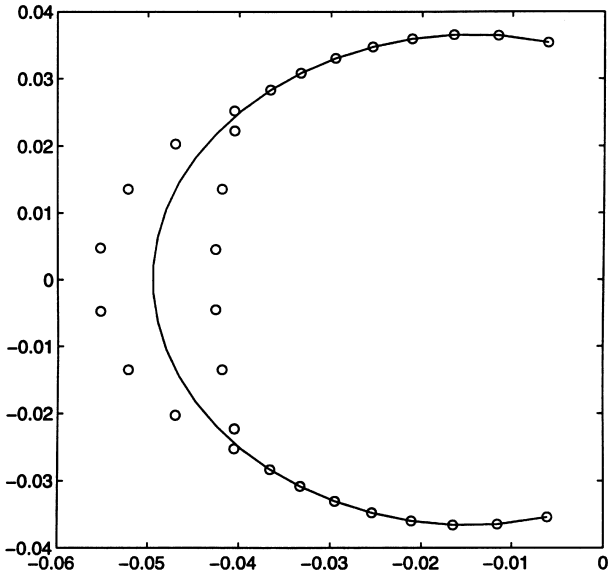

Fig. 1. Bifurcation of computed eigenvalues with respect to the right curve, for T_{30}.

It is well known that when eigenvalues are simple and well separated, but ill-conditioned, the matrix is very close to other ones with defective eigenvalues (e.g. [2–4,13,19,24,25]). The point is how close it is to the nearest defective matrix.

The aim of this paper is to provide a sharp estimate for such a distance when the considered matrix is T_n.

In Section 2 we present some basic results. In Section 3 we present different approaches to estimate the distance between T_n and the nearest defective matrix. Appendix A follows, reporting some new result about the eigenvalues of Bessel matrices.

The basic notation is as follows: $\mathbb{C}$ is the set of all complex numbers, $\mathbb{R}^+$ is the set of positive real numbers, $\mathbb{N}^+$ is the set of positive integer numbers, A^{T} is the transpose of matrix A, fl is the floating point operator, $k(A)$ is the condition number of matrix A, computed in 2-norm.

We have performed numerical tests mainly in MATLAB with IEEE double precision floating point arithmetic with unit roundoff $\varepsilon \approx 10^{-16}$.

2. Basic results

As already pointed out, the eigenvalues of matrix T_n are zeros of Bessel polynomials, which have been thoroughly studied. Their asymptotic properties and the location of these zeros were efficiently investigated and they are well known at present. We cannot attempt a survey of the vast literature. We just remark that this result [14] is particularly significant to our problem. There the author presents a general method for finding the zeros of a polynomial solution to second order linear homogeneous ordinary differential equations and shows how such a method is efficient for the classical Bessel polynomials.

Therefore the problem of accurately computing the eigenvalues of T_n has already been solved by means of a transformation of the numerical problem.

However if we consider T_n just as a test matrix for a case of (complex) ill-conditioned simple eigenvalues and we do not want to change the numerical problem, we have to resort to classical methods for the resolution of non-normal eigenproblem.

Several general methods were applied to find the eigenvalues of matrix T_n. They can be summed up as follows: QR algorithm (from LAPACK and MATLAB **eig**), DQR algorithm [21], BR algorithm [22], Schur decomposition (classical algorithm), Generalized Bairstow algorithm [10], Newton's method applied to the characteristic equation of the matrix T_n (using the three-term recurrence satisfied by the characteristic polynomial [23]), Newton's method applied to a suitable secular equations related to T_n [8,9], several types of polynomials zero-finding methods (among others [18]), several versions of

homotopy methods [16], Divide-and-conquer method [5], Arnoldi iteration method.

The graphical representations of eigenvalues $\lambda_i,\ i = 1, \ldots, n$, computed by all these methods overlap (with a few cases of extremely slight differences). Indeed all the methods exhibit the same behavior of the computed eigenvalues: when n becomes relatively large ($n \geqslant 25$ for double precision and $n > 12$ in single precision), in any graphical representation the eigenvalues remain simple (as they have to be), but they do not fit a well-defined curve (as they should do), because they exhibit a peculiar bifurcation, which remains the same for all the methods. As the resolution of MATLAB graphical representation is not high, we expect that computed eigenvalues exhibit really numerical instability even for some $n < 25$.

In order to compare the best known values Λ_i for eigenvalues, as computed by the algorithm presented in [14], with the eigenvalues λ_i of T_n computed by **eig** (which are essentially equal to the values computed by all the other methods), at first we compute $r_n = \min_i |\Lambda_i - \lambda_i|,\ i = 1, \ldots, n$, for different n.

Table 1 reports the (rounded) values of r_n which we have obtained.

We notice that there are always pairs of well-conditioned eigenvalues, which coincide with the exact ones and lie along the curve. Experimentally we have checked that the best conditioned eigenvalues are always those with the largest imaginary part (and their conjugates).

Then we want to compare $R_n = \max_i |\Lambda_i - \lambda_i|,\ i = 1, \ldots, n$. We recall that the eigenvalues lie along curves which quickly tend to the origin for increasing n. Therefore we can compare different R_n only if we "normalize" the eigenvalues so that the eigenvalue with the largest imaginary part becomes +i and its conjugate becomes −i. So all the eigenvalue curves coincide at least at the endpoints.

Table 2 reports the (rounded) values of R_n^* for such normalized eigenvalues Λ_i^* and $\lambda_i^*,\ i = 1, \ldots, n$.

It is clear that for any n large enough (say $n \geqslant 25$), there are pairs of ill-conditioned eigenvalues, which do not lie along the curve. Actually until

Table 1
Minimum of absolute differences between exact eigenvalues and computed eigenvalues

n	20	23	24	25	26	27	28	30	40	60
r_n	2e−15	4e−15	3e−15	2e−15	3e−15	2e−15	5e−15	4e−15	1e−15	3e−17

Table 2
Maximum of absolute differences between normalized exact eigenvalues and normalized computed eigenvalues

n	20	23	24	25	26	27	30	40	60
R_n^*	4.5e−6	2.6e−4	7.8e−4	2.1e−3	5.9e−3	1.3e−1	1.8e−1	7.2e−1	1.7e+0

$R_n^* < 1 \times 10^{-3}$, any graphical representation shows that Λ_i^* and λ_i^* coincide, even though the error cannot be considered small. Therefore if we accept a threshold of 5e−5 for R_n^*, we can say that for $n \geqslant 23$ some of the eigenvalues λ_i, $i = 1, \ldots n$, are not computed correctly any longer. Obviously if the accepted threshold decreases, n decreases as well.

For increasing n both the ill-conditioning and the number of ill-conditioned eigenvalues increase. Fig. 2 shows such a behavior for $n = 40$; there Λ_i^* are denoted by crosses and λ_i^* by circles.

We remark that for $n = 27$ there is a jump in R_n^* : the most ill-conditioned eigenvalue of T_{27} has no correct digits any longer.

We emphasize that *using a finite precision* $\varepsilon \approx 10^{-16}$, *the characteristic polynomial of* T_n $p(T_n) = \det(\lambda I - T_n)$ *remains* $< \varepsilon$ *in a region* (*around the curve of eigenvalues*) *which becomes larger and larger when n increases,* as we have experimentally checked. Therefore the computed eigenvalues of T_n, λ_i, $i = 1, \ldots, n$, can be considered actually eigenvalues of T_n *within the used precision*. However we think of them as "wrong" eigenvalues when they become so ill-conditioned that they do not lie along a well-defined curve as they should do.

One might think that the localization of eigenvalues affects their ill-conditioning. Actually they lie along curves which quickly tend to the origin. This means that they are getting more and more clustered for increasing n, even though they remain always simple. Then we have applied a conformal map

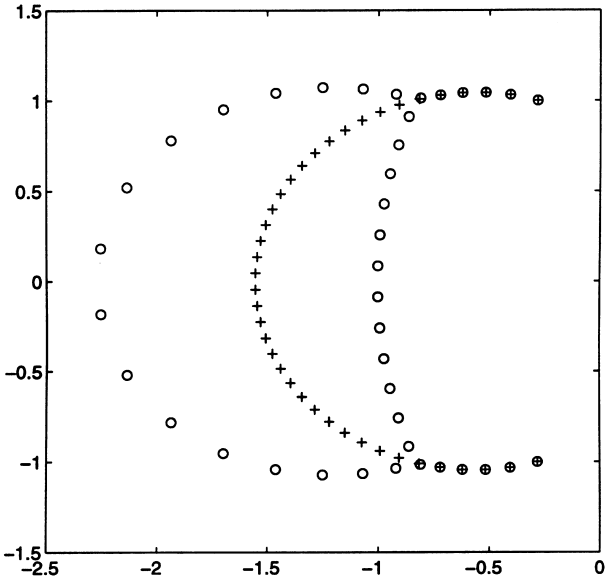

Fig. 2. Comparison between normalized exact eigenvalues (crosses) and normalized computed eigenvalues (circles), for T_{40}.

$w = Az + B$ (where A, B are two complex numbers and $z = e^{i\varphi}$) which corresponds to a rotation through the angle φ with a stretching or contraction by a factor $|A|$, followed by a translation through the vector displacement $|B|$ so that the problem $T_n x = \lambda x,\ n > 25$, becomes $T_n^t x = \lambda^t x$, where $T_n^t = wT_n,\ \lambda^t = w\lambda$. As the considered conformal mapping preserves similarity of geometrical figures, the eigenvalues λ^t will lie along a new similar but displaced curve in $\mathbb{C}$ and can be as far one from another as we want. Well, *the conditioning of the eigenvalues of the altered problem does not improve at all*.

The reason is that the set of eigenvectors remains the same.

We remark that the matrix X, built up by eigenvectors, exhibits a condition number increasing with n in such a way that for $n \geqslant 27,\ X$ becomes rank-deficient and T_n *is not diagonalizable any longer*, *despite the fact that it has always simple eigenvalues*. Actually for $n \geqslant 27,\ T_n$ behaves as a defective matrix, because its eigenvectors become linear dependent within the used precision ε (indeed the numerical rank determination depends on some small machine-dependent parameter ϵ), whereas the eigenvectors still correspond to n distinct eigenvalues of T_n.

We see that these results are in accordance with Table 2, where R_n^* exhibits a jump for $n = 27$.

However, even when X is only ill-conditioned, we can have troubles in eigenvalue calculations. This is enlightened by the following observation [11]. For any n, we have

$$\mathrm{fl}(X^{-1}T_nX) = X^{-1}T_nX + E,$$

where

$$\|E\|_2 \leqslant \varepsilon k(X)\,\|T_n\|_2.$$

As $\|T_n\|_2 \simeq 1.27$ for any n, then the upper bound for E varies with $k(X)$. When X is computed by means of MATLAB **eig**, we obtain the (rounded) values reported in Table 3.

It is clear that significant errors can be introduced into eigenvalue calculation when $k(X)$ is large with respect to the used precision ε; therefore we can expect some troubles even for some $n < 27$. Indeed for $n \geqslant 25$, as we already pointed out, the eigenvalues become so ill-conditioned that any graphical representation shows a peculiar bifurcation, whereas for $23 \leqslant n < 25$ only numerical instability appears (see Table 2).

Table 3
Condition number $k(X)$ of eigenvector matrix X

n	18	20	22	23	24	25	26
$k(X)$	3.4e+9	4.7e+10	6.5e+11	2.4e+12	9.1e+12	3.4e+13	1.3e+14

It is worth noting that for $n \geqslant 27$, even the eigenvectors corresponding to the exact eigenvalues $\Lambda_i,\ i = 1, \ldots, n$, are linearly dependent.

We recall that the matrix T_n can be seen as a skew-symmetric matrix S_n plus a rank-one matrix given by $e_1 e_1^{\mathrm{T}}$, where e_1 is the first standard unit vector. Thus T_n is a rank-one modification of a normal matrix. However in [17] it was shown that the classical departure from normality [12] fails to reflect the spectral instability of T_n for increasing n.

At last we remark that the eigenvalues of T_n lie along a curve symmetric with respect to the real axis; therefore the spectrum of T_n can be divided into three parts: all the eigenvalues along the curve above the real axis, the one real eigenvalue if n is odd, and all the eigenvalues along the curve below the real axis. If we continuously change non-zero entries in T_n, the number of eigenvalues in each part (i.e. the stable eigendecomposition) can change only when two eigenvalues coalesce on the real axis. As T_n, being unreduced tridiagonal, cannot have derogatory eigenvalues, its multiple eigenvalues are necessarily defective. This means that when $n \geqslant 27$ the matrix T_n is pathologically close to a defective matrix, the considered eigenproblem has to be viewed as numerically defective by the computer [2] and needs a larger machine precision for accurate computations.

Therefore the crucial task is how to measure the distance to the nearest defective matrix from $T_n,\ n < 27$.

3. The nearest defective matrix

In [17] the authors have already studied the behavior of the standard "individual condition numbers" ([11, Section 7.2.2]), of each eigenvalue of T_n.

Here we report the maximum $k_{\mathrm{M}}(n)$ of the individual condition numbers of the eigenvalues of T_n, which is always attained for the one real eigenvalue of T_n when n is odd, and for the complex conjugate pair of eigenvalues closest to the real axis when n is even (this means that those are the most ill-conditioned eigenvalues for any n); $k_{\mathrm{M}}^*(n)$ is analogous to $k_{\mathrm{M}}(n)$, but it is computed for the scaled matrix T_n^* whose eigenvalues are λ_i^* (see Table 2).

We remark that, as the condition numbers are computed by means of the left and right eigenvectors, they are obviously affected by the same sensitivity (see Table 3 about the ill-conditioning of right eigenvectors).

Moreover Table 4 shows that these individual condition numbers fail to give an account of the actual ill-conditioning of the eigenvalues. Actually the

Table 4
Maximum of eigenvalue condition numbers

n	23	24	25	26	27	30	40
$k_{\mathrm{M}}(n)$	3.0e+11	1.0e+12	3.9e+12	1.4e+13	4.3e+13	6.4e+13	3.9e+13
$k_{\mathrm{M}}^*(n)$	3.0e+11	1.0e+12	3.9e+12	1.4e+13	5.2e+13	4.2e+13	7.9e+13

condition number is infinite for defective eigenvalues. We have shown (see Table 3) that T_n, $n \geqslant 27$, is defective within the used machine precision ε. Therefore Table 4 should show any jump for $n \geqslant 27$, analogously to Table 2. It is clear that this does not happen. Even though we want to use relative measures, results do not change, as $\|T_n\|_2 \simeq 1.27$ and $\left\|T_n^*\right\|_2 \approx 10$ for any n in Table 4.

Therefore for T_n the distance to the nearest defective matrix cannot be estimated in terms of the reciprocal of the largest individual condition number, as usually done.

In order to effectively measure the distance from T_n to the nearest defective matrix, we define the following parameter:

$$d(T_n) = \min\{\|\Delta\|_2 \mid \Delta \in \mathbb{C}^{n\times n},\ T_n + \Delta \text{ is } \textit{defective}\}.$$

This parameter was estimated and discussed in [13,24,25] and the references therein.

We remark that a necessary condition for any matrix to be defective is that it have at least one multiple eigenvalue. In general this condition is nearly, not quite sufficient for defectiveness (almost all matrices with multiple eigenvalues are defective). In our case this condition is also sufficient only if $T_n + \Delta$ is still unreduced tridiagonal, as we have already pointed out.

Clearly $d(T_n) < \varepsilon$, for $n \geqslant 27$.

The main result of this section is a very sharp bound for $d(T_n)$, $n < 27$, which we obtained by means of two different approaches.

Nevertheless we will not give any constructive description of the matrix Δ of minimal 2-norm such that $T_n + \Delta$ is defective.

3.1. SVD approach

Here we tackle the problem of finding a close defective matrix with a given eigenvalue λ which is at least double.

It is well known that to study whether a matrix $A = U\Sigma V^{\mathrm{T}}$ of rank r is close to a matrix of lower rank, one can look at its smallest singular value σ_r. If this value is very small, then the matrix is very close to a matrix of rank $r-1$. Indeed in this case there exists a perturbation of norm σ_r which produces a matrix of rank $r-1$: one such perturbation is actually provided by $\sigma_r u_r v_r^{\mathrm{T}}$, where u_r and v_r are the left and right singular vectors. Moreover the smallest singular value of A provides the 2-norm distance of A to the set of all rank-deficient matrices.

For our purpose, we can use these results for matrices T_n in the following way.

Proposition 1. *Let A be an $n \times n$ matrix; λ_1 and λ_2 a pair of close eigenvalues (either real or complex conjugates) and $l = mean(\lambda_1, \lambda_2)$. Set $B = A - lI = U\Sigma V^{\mathrm{T}}$. Then the perturbation $M = \sigma_n u_n v_n^{\mathrm{T}}$ produces a matrix $N = A - M$ which has l as a real eigenvalue.*

Proof. Obviously the matrix $B = A - lI$ is a matrix of rank n, as l is not an eigenvalue of A. Then in the SVD decomposition $B = U\Sigma V^{\mathrm{T}}$ the smallest singular value is $\sigma_n \neq 0$. Therefore the perturbation $M = \sigma_n u_n v_n^{\mathrm{T}}$ produces a matrix $H = B - M$ which is of rank $n-1$. Actually we have $H = B - M = A - lI - M = (A - M) - lI = N - lI$, hence l is an eigenvalue of $N = A - M$. □

Remark 1. The same result holds when $N = A + M,\ H = B + M$.

Consider $A = T_n,\ n$ even; λ_1 and λ_2 the closest complex conjugate eigenvalues.

Using Proposition 1 we can build a very close matrix N with a real eigenvalue l which is not necessarily double. Only in the case that σ_n is large enough with respect to ε, we can use the following MATLAB loop [6] in order to find a close matrix N with a double real eigenvalue:

while convergence condition is not satisfied
  $[u,s,v] = \mathrm{svd}(a - l^{*}\mathrm{eye}(\mathrm{size}(a)))$
  $n = a - u(:,n)^{*}s(n,n)^{*}v(:,n)^{\mathrm{T}}$
  $l =$ mean of two close real eigenvalues of matrix n
end

Example 2. Let $n = 14$; the pair of closest complex eigenvalues of T_n is given by (rounded)

$$\lambda_{1,2} = -1.03501041\mathrm{e}-1 \pm 9.37153188\mathrm{e}-3i;$$

then

$$l = -1.03501041\mathrm{e}-1$$

and

$$\sigma_n = 1.8\mathrm{e}-9.$$

We set as a convergence condition that the absolute value of difference between two close real eigenvalues is less than tol. $= 5\mathrm{e}-7$. Then we obtain a final matrix N with a double real eigenvalue $l = -1.03037\mathrm{e}-1$. Unfortunately this matrix N is not defective.

In any case we recall that *our aim is just the evaluation of the distance to the closest defective matrix, not its construction.*

When σ_n is small enough with respect to the machine precision ε (say $\sigma_n < \varepsilon \cdot n \cdot \|T_n\|_2$, according to the MATLAB rank evaluation), then $B = T_n - lI$ is rank-deficient; therefore l becomes an eigenvalue of T_n. However, as $l = \text{mean}(\lambda_1, \lambda_2)$, this can happen only when $l = \lambda_1 = \lambda_2$; this means that λ_1 and λ_2 are to be considered equal to l within the used machine precision ε. Hence eigenvalue l is to be considered double and therefore defective, as already pointed out.

We remark that when n is odd, the behavior of the real eigenvalue l of T_n is hard to define; indeed it can be either multiple or simple (in this second case the other previous real eigenvalue splits into two complex conjugate ones).

Therefore according to our definition of $d(T_n)$ we can use the SVD approach only for even n.

Table 5 reports our (rounded) results.

For even $n \geqslant 26$, $\sigma_n < \varepsilon$, the machine precision. From Table 5 it is clear that for $n \geqslant 24$ *T_n is to be considered a defective matrix and its closest eigenvalues are seen as the same double eigenvalue within the machine precision, even though they are always simple from a theoretical point of view*.

This SVD approach is inspired by Algorithm 1 [6], but it is originally tailored to our matrix T_n.

We compared our results with those that one can obtain by means of MATLAB template presented in [7]. The template code **sg_min** used with convergence tolerances equal to $1e-15$, converges only for $n \geqslant 24$ in order to find the nearest defective matrix (with a particular Jordan structure): this means that our results completely agree with those. We emphasize that both methods find the distance to the nearest defective matrix with a given double eigenvalue; however in general there may be closer defective matrices with multiple eigenvalues different from those we imposed, as it was pointed out by Wilkinson [24]. Therefore we expect that there exists some close defective matrix with a double eigenvalue slightly different from that we found.

We point out that Tables 4 and 5 cannot be compared, as they refer to measures of different parameters.

3.2. Statistical approach

Here we tackle the problem of finding a close defective matrix with any multiple eigenvalue.

Table 5
Minimal singular value of matrix B

n	σ_n	n	σ_n	n	σ_n
2	7.7e−2	10	5.1e−7	18	7.2e−12
4	3.3e−3	12	3.0e−8	20	4.6e−13
6	1.6e−4	14	1.8e−9	22	3.0e−14
8	8.9e−6	16	1.1e−10	24	1.9e−15

To this end we have decided to resort to a statistical approach. In particular we have followed the method proposed by Chaitin-Chatelin and Frayssé [2]. These authors presented the toolbox PRECISE, which allows numerical experiments by a random normwise perturbation of a given matrix A which uses:

- a random variable μ taking the value ± 1 with equal probability;
- a parameter t, which controls the size of the perturbations; in double precision (mantissa of 53 bits), t is chosen as 2^{-p}, $p \in \mathbb{N}^+$, $0 \leqslant p \leqslant 52$, so that $\varepsilon \leqslant t \leqslant 1$, where ε is the machine precision;
- a matrix ΔA which is added to A, such that $\Delta A = \|A\|_2 tF$, where F is a random matrix with entries equal to μ.

Let $(\lambda_\theta, x_\theta)$ be a fixed eigenpair for T_n computed by MATLAB **eig**. Then the associated residual is $Ax_\theta - \lambda_\theta x_\theta$.

The normwise backward error is $B(x_\theta) = \|Ax_\theta - \lambda_\theta x_\theta\|_2 \|A\|_2 \|x_\theta\|_2$.

For each size t of perturbations, we compute a sample of eigenvalues Λ_t (mean m_t, standard deviation γ_t), eigenvectors X_t, and a sample of residuals $\mathrm{r}_t = Ax_\theta - \lambda_\theta x_\theta$ (mean ρ_t, standard deviation v_t). The means m_t and ρ_t can be real or complex.

The normwise statistical estimation of the individual condition number for the given simple eigenvalue is computed [2] as:

$$K_t = \frac{\gamma_t}{|\lambda_\theta|} \left(\frac{\|A\|_2 \, \|x_\theta\|_2}{\|v_t\|_2} \right). \tag{2}$$

The corresponding error estimation is given by $\mathrm{Er}_t = K_t \cdot B(x_\theta)$.

Obviously (2) is affected by the same sensitivity as the eigenvectors and the eigenvalues. However we remark that the statistical approach uses a qualitative point of view. In practice we let each entry in matrix A "vibrate" many times around its original value with a given size t of perturbations; then for each t we compute a statistical estimation of K_t and Er_t. This way we extract relevant information about the distance to the nearest defective matrix from many possibly incorrect results.

About the reliability of these statistical estimates, we remark that the map $t \to I_t = B(X_t)t$ is constant whenever the backward error at X_t is of order of 1 in t. The interval on which the map I_t is constant and such that $I_t \sim 1$, is called *the reliability interval* [2].

In all our tests the reliability interval coincides with the interval $[\varepsilon, 1]$, hence the condition of backward stability is satisfied [2] within the machine precision ε.

We remark that we have used 2-norm instead of the ∞-norm as in [2], in order to obtain comparisons with our previous results.

Indeed the map $t \to K_t$ exhibits a typical behavior. While t increases from 10^{-15} to 10^0, K_t remains constant for a while, then it shows a maximum and afterwards it decreases. The value t_0 for which K_t ceases to be constant provides

the experimental value of a *dangerous border*, that is the estimated maximum perturbation that the matrix can bear in order to remain non-defective. Obviously when the considered eigenvalue becomes defective, its individual condition number becomes infinite, but usually the family of chosen random perturbations of size t does not completely contain the specific perturbation that achieves numerical singularity; this is the reason why in our examples we have checked a significant maximum for K_t but not a very high one, whereas it should be infinite.

As an example, in Table 6 we report the results referring to matrix T_{16}.

We remark that for $t \leqslant 3.6\text{e}-12$, K_t is to be considered constant in the statistical sense. It is clear that in this case K_t achieves its infinite value for $3.6\text{e}-12 < t < 7.3\text{e}-12$, whereas we obtain only a maximum for $t = 7.3\text{e}-12$. From that we assume as the *dangerous border* the value $t_0 \approx 7.3\text{e}-12$.

We can interpret t_0 as an estimate of the distance of T_n to the nearest defective matrix, viewed by the computer in a statistical way: we call such a distance $\mathrm{ds}(T_n)$.

We emphasize that t can assume only a discrete set of values, as was pointed out before, hence the computation of $t_0 = \mathrm{ds}(T_n)$ can only be approximative.

When the machine precision is ε, ([2, p. 121]) three cases can happen:

1. $\mathrm{ds}(T_n) \geqslant 1$: the considered eigenvalue is so well-conditioned that even high relative perturbations do not affect it;
2. $\mathrm{ds}(T_n) \in [\varepsilon, 1]$: is the typical case;
3. $\mathrm{ds}(T_n) \leqslant \varepsilon$: the eigenvalue is so ill-conditioned that a higher machine precision is needed.

Table 7 reports our (rounded) results, relating to the eigenvalues of T_n closest to the real axis.

Table 6
Statistical estimation of condition number of the worst conditioned eigenvalue, for T_{16}

t	K_t	t	K_t	t	K_t	t	K_t
1.8e−15	5.4e+8	1.4e−11	4.3e+9	6.0e−8	2.9e+6	2.4e−4	2.6e+3
3.5e−15	6.3e+8	2.9e−11	2.6e+9	1.2e−7	1.4e+6	4.8e−4	1.3e+3
7.1e−15	4.2e+8	5.8e−11	1.3e+9	2.4e−7	7.9e+5	9.8e−4	8.3e+2
1.4e−14	5.0e+8	1.2e−10	8.8e+8	4.8e−7	4.4e+5	1.9e−3	5.1e+2
2.8e−14	6.4e+8	2.3e−10	4.2e+8	9.5e−7	2.2e+5	3.9e−3	2.8e+2
5.7e−14	3.5e+8	4.6e−10	2.2e+8	1.9e−6	1.4e+5	7.8e−3	1.8e+2
1.1e−13	6.3e+8	9.3e−10	1.2e+8	3.8e−6	6.8e+4	1.6e−2	9.9e+1
2.3e−13	5.1e+8	1.9e−9	6.6e+7	7.6e−6	4.0e+4	3.1e−2	6.8e+1
9.1e−13	5.2e+8	3.7e−9	3.1e+7	1.5e−5	2.2e+4	6.2e−2	5.7e+1
1.8e−12	5.9e+8	7.4e−9	1.6e+7	3.0e−5	1.3e+4	1.2e−1	4.7e+1
3.6e−12	4.3e+8	1.5e−8	1.0e+7	6.1e−5	7.1e+3	2.5e−1	3.9e+1
7.3e−12	5.0e+9	3.0e−8	5.3e+6	1.2e−4	4.0e+3	–	–

Table 7
Dangerous border

n	$\mathrm{ds}(T_n)$	n	$\mathrm{ds}(T_n)$	n	$\mathrm{ds}(T_n)$
2	6.2e−2	10	1.2e−7	18	9.1e−13
3	3.9e−3	11	3.7e−9	19	5.7e−14
4	3.9e−3	12	3.7e−9	20	5.7e−14
5	6.1e−5	13	1.2e−10	21	3.5-15
6	6.1e−5	14	4.6e−10	22	3.5e−15
7	1.9e−6	15	1.4e−11	23	$<\varepsilon$
8	1.9e−6	16	7.3e−12	–	–
9	6.0e−8	17	9.1e−13	–	–

We remark that usually no significant difference is shown between the values of $\mathrm{ds}(T_n)$ for an odd n and the subsequent even $n+1$, $n \geqslant 3$.

From Table 7 we see that for $n \geqslant 23$, T_n is to be viewed as a defective matrix by a computer using double precision: *this is in accordance with our previous results* (see Sections 2 and 3.1). Indeed in the graphical representation the bifurcation appears for $n \geqslant 25$, however the numerical ill-conditioning appears even for smaller n, as we have already pointed out.

4. Conclusions

We have studied the behavior of eigenvalues of the Bessel matrix T_n, $n \in \mathbb{N}^+$, which are always simple, but in practice become defective using finite precision.

To this end we have tackled the problem of computing the distance from the Bessel matrix T_n, to the nearest defective matrix, using two different approaches. They differ, as the first one provides an estimation of the distance to the nearest defective matrix with an imposed multiple eigenvalue; whereas the second one looks for the closest defective matrix with any multiple eigenvalue; moreover the second approach is of statistical type. We have shown that the second approach provides a little sharper estimation of the distance to the nearest defective matrix, as it was expected. We conclude that the reason why a numerical instability appears for $n \geqslant 23$ (in double precision arithmetic) is that the matrix T_n becomes so close to a defective matrix that it can be considered numerically defective itself (within the machine precision).

It is clear that whenever we use a machine precision different from double one, the threshold $n = 23$ changes.

Acknowledgements

The author is very indebted to Prof. F. Uhlig, who provided useful discussions, and to the referees for their concerned and accurate revising.

Appendix A. Something more about the eigenvalues of Bessel matrices

As we have already emphasized (Section 2, Table 1) there are always pairs of well-conditioned eigenvalues which can be computed exactly by means of classical methods already mentioned. If we do not want to lose these information, we can exploit them in order to find an approximation of the curve of exact eigenvalues. We suggest to use the Limaçon of Pascal

$$r = b - a \cos \vartheta. \tag{A.1}$$

It has to be considered in the left-half of plane, when we think of Gaussian plane as the usual Cartesian plane with polar coordinates. As (A.1) depends on two parameters, only a few exact eigenvalues suffice to approximate a, b in the sense of least-squares approximation.

For example we consider the case $n = 60$. Using only four pairs of complex conjugate eigenvalues with the largest imaginary part, we found $a = 1.6573\text{e}-2,\ b = 8.994\text{e}-3$. Fig. 3 shows our results: the computed eigenvalues are indicated by circles, the curve of exact eigenvalues is indicated by solid line, the approximated curve is indicated by dotted line. If we call $\overline{\Lambda}_i$ the value on the approximated curve, corresponding to the exact eigenvalue $\Lambda_i,\ i = 1, 2 \ldots, 60$, we obtain

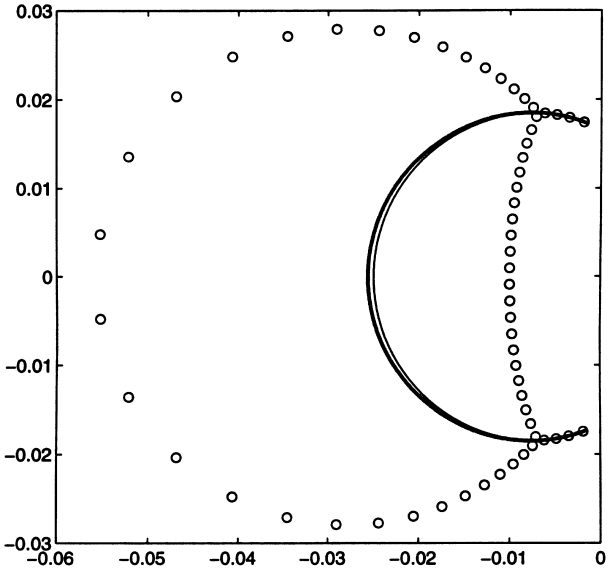

Fig. 3. Comparison among computed eigenvalues (circles), curve of exact eigenvalues (inside solid line), approximated curve (outside dotted line).

$$\max_i \left|\Lambda_i - \overline{\Lambda}_i\right| \simeq 6.3\text{e}-4, \quad \min_i \left|\Lambda_i - \overline{\Lambda}_i\right| \simeq 3.6\text{e}-8.$$

This behavior remains the same even for the other large n we tried.

Therefore we can consider the proposed curve as a good approximation of the exact one, which we emphasize *is not known explicitly* [1].

Moreover the proposed curve can be used effectively in order to build a better starting point vector for the numerical problem in [14].

At last we remark that each point of the curve (A.1) is an eigenvalue of T_{60} in the sense that it let the characteristic polynomial be null (see Section 2). Moreover [1, p. 477] it is known that for large n the eigenvalues are roughly uniformly distributed in angle in the left-half of the plane. So *exploiting a few correct eigenvalues which can be calculated exactly by means of classical methods, one can obtain good approximations of both the correct eigenvalue curve and all the exact eigenvalues.*

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